Topics covered:
Relations, Functions, Bases and Number Systems, Computer representation of numbers, BCD addition and ASCII codes.

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Question 1

Consider the functions f, g and h, all defined on the set {0, 1, 2, 3, …, 12}

x	0	1	2	3	4	5	6	7	8	9	10	11	12
f(x)	8	11	5	9	4	2	12	7	0	3	10	1	6

x	0	1	2	3	4	5	6	7	8	9	10	11	12
g(x)	10	5	12	1	9	6	0	7	11	2	3	8	4

x	0	1	2	3	4	5	6	7	8	9	10	11	12
h(x)	10	11	12	6	5	4	0	1	2	3	9	8	7

(i)   Write down the values of: $g(h(f(8)))$ and $f^{-1}(g^{-1}(0))$

(ii)   Construct a table of values (like those shown above) for $h(g^{-1}(x))$.

(iii)   Construct a table for $f(f(x))$. What can you conclude about the inverse of f?

(iv)   Construct a table for $h^{-1}(x)$, and draw its graph on the grid provided on the last page.

[5 marks]

Question 2

Suppose there is a set of growers G = {a, b, c, d}, a set of retailers R = {e, f, g} and a set of customers C = {m, n, p, q, r}. There are two relations A and B on G × R and R × C, respectively, defined by:

aAf, bAe, bAg, cAg, dAf,   and   eBn, fBp, fBq, gBm, gBr

xAy means "grower x sold goods to retailer y", and
$yA^{-1}x$ means "retailer y bought goods from grower x"

xBy means "retailer x sold goods to customer y", and
$yB^{-1}x$ means "customer y bought goods from retailer x"

(i)   Find the matrices M(A) and M(B) that represent the relations A and B.

(ii)   Find the matrices $M(A)^T$ and $M(B)^T$ that represent the relations A^{-1} and B^{-1}

(iii)   Consider the query:

**Which customers have received goods that came from
the same grower(s) as those goods received by customer p?**

Find the logical matrix products M(A) M(B) and then $M(B)^T M(A)^T$, and finally $M(B)^T M(A)^T M(A) M(B)$, and hence answer the query.

[Hint: To see a similar exercise, look at the "Application" on page 13 of Lecture 5.]

[1 + 1 + 3 = 5 marks]

Question 3

(a)   Consider the following table:

decimal	octal	binary	hexadecimal
73 →	→	→	→ ↓
125 →	→	→	→
←	←	←	↓

(i)   Convert each of the decimal numbers in the first column to octal.
(ii)   Convert the two octal numbers to binary.
(iii)   Convert the two binary numbers to hexadecimal.
(iv)   Add the two hexadecimal numbers.
(v)   Convert the hexadecimal sum to binary, then to octal and then to decimal.
(Give your answers in a table like that above.)

(b)   (i)   Convert the decimal fraction 0.296875 to binary;
        (ii)   Convert the decimal fraction 0.453125 to binary;
        (iii)   Add the two binary fractions from (i) and (ii);
        (iv)   Convert the binary fraction from (iii) to decimal.
(Show your working)

(c)   Add the following, given that (i) is binary, (ii) is octal and (iii) is hexadecimal:

(i)   1110101   (ii)   2765   (iii)   2FE58
        1010111 +   3616 +   1EE4C +
        ─────────   ──────   ────────

(d)   Suppose that the following numbers are all hexadecimal. Carry out the additions with the appropriate "decimal adjustments" so as to obtain answers that are correct when interpreted as decimal. (Show all your working)

(i)   27495   (ii)   156290
        33232 +   285612 +
        ────────   ─────────

[5 + 2 + 3 + 2 = 12 marks]

Question 4

For each of the following, suppose that two 8-bit binary numbers have been added. In each case the 8-bit output is given and the values of the N, V and C flags. For each case give the correct answer as a decimal number:

      (a)   if the result is interpreted as the sum of **unsigned** integers;

      (b)   if the result is interpreted as the sum of **signed** integers.

	8-bit output	N	V	C
(i)	0011 1000	0	0	1
(ii)	0100 0110	0	1	1
(iii)	1011 1101	1	0	0
(iv)	1011 1001	1	0	1
(v)	1100 1101	1	1	0

[10 × 1 = 10 marks]

Question 5

Introduction – Part 1: Two functions

This question uses two types of function that you may not have met before, the CEILING function CLG(x) and the MOD function MOD(x, n).

Definition of the CEILING function:

CLG(x) is the smallest integer that is greater than or equal to x. Or, if you prefer, it is the smallest integer that is not smaller than x.

      eg.   CLG(4.3) = ?

           Which integers are not smaller than 4.3?
           Answer: 5, 6, 7, …, etc
           Of these, which is the smallest?
           Answer: 5
           So CLG(4.3) = 5

      eg.   CLG(7.5) = ?

           Which integers are not smaller than 7.5?
           Answer: 8, 9, 10, …, etc
           Of these, which is the smallest?
           Answer: 8
           So CLG(7.5) = 8

      eg.   CLG(6) = ?

           Which integers are not smaller than 6?
           Answer: 6, 7, 8, …, etc
           Of these, which is the smallest?
           Answer: 6
           So CLG(6) = 6

Practice exercises:

(i)	CLG(3.8) = …………..	(ii)	CLG(12.6) = …………….
(iii)	CLG(11) = …………..	(iv)	CLG(13.7) = …………….
(v)	CLG(5) = …………..	(vi)	CLG(6.9) = …………….
(vii)	CLG(5.7) = …………..	(viii)	CLG(15.6) = …………….

Answers: (i) 4;   (ii) 13;   (iii) 11;   (iv) 14;   (v) 5;   (vi) 7;   (vii) 6;   (viii) 16

Definition of the MOD function:

MOD(x, n) is the remainder when x is divided by the whole number n.

eg.   MOD(38, 5) = ?
When 38 is divided by 5, what is the remainder?
Answer: 3
So, MOD(38, 5) = 3

eg.   MOD(29, 8) = ?
When 29 is divided by 8, what is the remainder?
Answer: 5
So, MOD(29, 8) = 5

eg.   MOD(13, 4) = ?
When 13 is divided by 4, what is the remainder?
Answer: 1
So, MOD(13, 4) = 1

eg.   MOD(6, 10) = ?
When 6 is divided by 10, what is the remainder?
Answer: 6
So, MOD(6, 10) = 6

eg.   MOD(47, 12) = ?
When 47 is divided by 12, what is the remainder?
Answer: 11
So, MOD(47, 12) = 11

Practice exercises:

(i)	MOD(17, 5) = …………..	(ii)	MOD(75, 12) = …………….
(iii)	MOD(36, 9) = …………..	(iv)	MOD(45, 10) = …………….
(v)	MOD(3, 8) = …………..	(vi)	MOD(29, 6) = …………….

(vii)   MOD(24, 4) = …………..   (viii)   MOD(55, 7) = ……………

(ix)   MOD(21, 5) = …………..   (x)   MOD(5, 7) = ……………

Answers: (i) 2;   (ii) 3;   (iii) 0;   (iv) 5;   (v) 3;   (vi) 5;   (vii) 0;   (viii) 6;   (ix) 1;   (x) 5

Introduction – Part 2: Application to Computer Science

Computer scientists often use **matrices** or rectangular **arrays** of numbers. Although they may appear on a computer monitor as a rectangular array, they are usually stored in memory locations as linear **sequences** of numbers. As a very simple case, suppose we have an array with 5 columns, such as:

18	12	5	16	13
6	7	21	20	11
22	10	4	8	16
2	19	11	21	12
9	1	19	10	5
14	9	17	11	19
4	3	21	9	10

and we want to store the array in a linear sequence starting with the first row, then the second and so on. So the sequence of stored values would begin:

   18   12   5   16   13   6   7   21   20   11   22   10   4   8   16   2   19   11   … etc

There are two important types of questions concerning the storage and retrieval of such a set of numbers:

   Type A: Given the row number and column number of an element in the array, how to determine its position in the linear sequence?

   Type B: Given the position of an element in the linear sequence, how to determine its row number and column number in the array?

There are formulae that can be used to answer such questions, and they make use of the CEILING and MOD functions. They also use the variables:

      num_cols — the number of columns in the array
      row_num — the row number of the element
      col_num — the column number of the element
      seq_pos — the position of the element in the linear sequence

Here are the formulae that provide the answers:

$$\text{seq\_pos} = (\text{row\_num} – 1) \times \text{num\_cols} + \text{col\_num} \qquad \text{(answer to Type A)}$$

$$\left.\begin{array}{l} \text{row\_num} \;=\; \text{CLG(seq\_pos / num\_cols)} \\[4mm] \text{col\_num} \;=\; \text{MOD(seq\_pos} - 1, \text{num\_cols}) + 1 \end{array}\right\}$$ (answers to Type B)

For example, consider the element in the 3rd row and 4th column of the array given above (which happens to be an 8). Notice also that the array has 5 columns.

The Type A question is: Given that the element is in the 3rd row and 4th column, what is its position in the linear sequence?

Answer:

$$\begin{aligned} \text{seq\_pos} \;&=\; (\text{row\_num} - 1) \times \text{num\_cols} \;+\; \text{col\_num} \\ &=\; (3 - 1) \times 5 \;+\; 4 \;=\; 14 \end{aligned}$$

So the element is the 14th in the linear sequence (which is obviously true.)

The Type B question is: Given that the element is in the 14th position in the linear sequence, what are its row and column numbers in the array?

Answer:

$$\begin{aligned} \text{row\_num} \;&=\; \text{CLG(seq\_pos / num\_cols)} \\ &=\; \text{CLG}(14 / 5) \;=\; \text{CLG}(2.8) \;=\; 3 \\ \text{col\_num} \;&=\; \text{MOD(seq\_pos} - 1, \text{num\_cols}) + 1 \\ &=\; \text{MOD}(14 - 1, 5) + 1 \;=\; \text{MOD}(13, 5) + 1 \;=\; 3 + 1 \;=\; 4 \end{aligned}$$

So the element is in the 3rd row and 4th column.

Practice examples: Fill in the blanks in the table

	Number of columns in array	Row number	Column number	Sequence position
(i)	8	5	3	
(ii)	12	6	2	
(iii)	15			50
(iv)	30			200

Answers: (i) 35;   (ii) 62;   (iii) 4, 5;   (iv) 7, 20

Your task for Question 5

First work through the above examples **in detail** to make sure that you understand the   process.
Then calculate the missing entries for the seven exercises in the table below.
(Give your answers in a table like the one below.)

	Number of columns in array	Row number	Column number	Sequence position
(i)	14	5	11	
(ii)	36	1	27	
(iii)	9	4	8	
(iv)	28	17	11	
(v)	30			98
(vi)	45			300
(vii)	24			85

[8 marks]

$f^{-1}(x)$

12

10

8

6

4

2

0

0   2   4   6   8   10   12   x